U291. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a bounded function and let $\mathcal{S}$ be the set of all increasing maps $\varphi\colon \mathbb{R} \to \mathbb{R}$. Prove that there is a unique function g in $\mathcal{S}$ satisfying the conditions

a) $f(x) \le g(x)$ for all $x \in \mathbb{R}$.

b) If $h \in \mathcal{S}$ and $f(x) \le h(x)$ for all $x \in \mathbb{R}$ then $g(x) \le h(x)$ for all $x \in \mathbb{R}$.

Proposed by Marius Cavachi, Constanta, Romania

Solution by Arkady Alt, San Jose, California, USA

a) Since f is bounded then for any $x \in \mathbb{R}$ set $G(x) := \{f(t) \mid t \in \mathbb{R} \text{ and } t \le x\}$ is bounded. Therefore for any $x \in \mathbb{R}$ we can define $g(x) := \sup G(x)$ and, obviously, that function $g(x)$ defined by such way satisfy to condition (a).

b) Let now $h \in S$ and $f(x) \le h(x)$ for all $x \in \mathbb{R}$.
Since $f(t) \le h(t)$ for any $t \le x$ then $g(x) = \sup\limits_{t \le x} f(t) \le \sup\limits_{t \le x} h(t) = h(x)$ (since h is increasing in $t \in (-\infty, x]$).

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy.